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Bayesian Prediction of Future Order Statistics and k-Record Values based on Progressive Type-II Censored Sample for Generalized Exponential distribution

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ABSTRACT

In this paper, we consider the problem of two-sample Bayesian prediction of order statistics and lower k-record values from a future sample asing from a generalized exponential distribution based on progressively type-II censored sample. We obtain the Bayesian point predictors based on squared error and linex loss functions. It is observed that the predictors cannot be obtained in closed forms. So we prpose importance sampling method to obtain the estimators for both point and interval predictors.

Key words and Phrases: *Generalized exponential distribution, progressively type-II censoring, order statistics, lower* k*-record values, Bayesian prediction, importance sampling method.*

1 Introduction

In many life-testing and reliability studies, the experimenter may not be able to gather complete information on failure times of all experimental units placed on a life-test. In medical or industrial applications, researchers have to treat the censored data because they usually do not have sufficient time to observe the lifetime of all subjects in the study. In some circumstances, the removal of units before they fail is planned in advance to reduce the cost and time associated with the experiment. Data associated to such

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experiments are called censored data. The type-I and type-II censoring are the most popular censoring techniques. The traditional type-I and type-II censoring techniques lack the flexibility of removal of items at points other than the terminal points of the experiment. For this reason, Cohen (1963) considered a more general censoring method called progressive type-II right censoring. Under this censoring, n identical units are placed on a life test at time zero. Immediately following the first failure, R_1 surviving units are removed from the test at random. Then immediately following the second failure, R_2 surviving units are removed from the test at random. This process continues until, at the time of the mth observed failure, the remaining $R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m$ units are all removed from the experiment. Here $R_1, R_2, \ldots, R_m$ are called censoring scheme and are pre-fixed.

Consider an experiment with n identical units being placed on a life test and suppose $X_1, X_2, \ldots, X_n$ denote their corresponding lifetimes of the units which are distributed according to a continuous population with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Let $(R_1, R_2, \ldots, R_m)$ be the pre-fixed censoring scheme and let $X_{1:m:n}, X_{2:m:n,} \ldots, X_{m:m:n}$ be the failure times of m units. The m ordered failure lifetimes $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ are called progressively type-II right censored order statistics of size m. The joint pdf of $X_{1:m:n}, X_{2:m:n,}, \ldots, X_{m:m:n}$ is given by (see, Balakrishnan and Aggarwala, 2000)

$$f_{m:n}(x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n}) = C \prod_{i=1}^{m} f(x_{i:m:n}) \left[1 - F(x_{i:m:n})\right]^{R_i}, \tag{1.1}$$

where $x_{1:m:n}, x_{2:m:n}, \ldots, x_{s:m:n}$ is the observed realization of $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ and

$$C = n\,(n - R_1 - 1)\,(n - R_1 - R_2 - 2))\ldots(n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)\,. \tag{1.2}$$

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n arising from a population with absolutely continuous cdf $F(x)$ and pdf $f(x)$. If we arrange the observations $X_1, X_2, \ldots, X_n$ in an increasing order of magnitude as $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, the order statistics of the sample are obtained. Then $X_{i:n}$ is called the ith order statistic. Order statistics can be found in a wide range of applications, including industry, reliability analysis, and material strength. For further discussions regarding the order statistics, one may refer to Arnold et al. (2008) and David and Nagaraja (2004). One major application of order statistics in the study of reliability of a system is the following. A system is called a $k-out-of-m$ system if it consists of m components and the system functions satisfactorily if at least $k(\leq m)$ components function. If the lifetimes of the components

are independently distributed, then the lifetime of the system coincides with that of the $(m-k+1)$ th order statistic of the life times of the components. Thus, order statistics plays a key role in studying the lifetimes of such systems. The cdf of $X_{i:n}$ is given by (see, Arnold et al., 2008)

$$F_{i:n}(x) = \sum_{r=i}^{n} \binom{n}{r} [F(x)]^r [1-F(x)]^{n-r}. \tag{1.3}$$

The pdf corresponds to the cdf (1.3) is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} [F(x)]^{i-1} [1-F(x)]^{n-i} f(x), \tag{1.4}$$

where B $(\cdot,\cdot)$ denote the complete beta function.

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables having an absolutely continuous cdf $F(x)$ and pdf $f(x)$. An observation X_j is called a lower record if $X_j < X_i$ for every $i < j$. An analogous definition deals with upper record values. Records of iid random variables and their properties have been studied extensively in the literature. For more details, one may refer to Arnold et.al (1998).

One of the difficulties that one may face while handling problems related to inferece with record data is that the expected waiting time for occuring subsequent records after first is infinite. Such an issue does not arise if we consider the k-records suggested by Dziubdziela and Kopocinski (1976). We use a formal definition of k-record values given by Arnold et al. (1998).

For a fixed but arbitrary positive integer k, the nth lower k-record times $T_{n(k)}$ and the nth lower k-record values $L_{n(k)}$ are defined as follows:

$$T_{1(k)} = k$$

and for $n > 1$

$$T_{n(k)} = min\left\{j : j > T_{n-1(k)}, X_j < X_{T_{n-1(k)}}\right\},$$

where $X_{i:m}$ denotes the ith order statistic in a sample of size m. The sequence of lower k-records are then defined by for $n \geq 1$

$$L_{n(k)} = X_{k:T_{n(k)}},\ k \geq 1.$$

In an analogous way, one can define the upper k-record values. The ordinary record values are contained in the k-records as a special case of $k = 1$.

The pdf of the nth lower k-record value $L_{n(k)}$ is given by (see Arnold et.al, 1998)

$$f_{n(k)}(x) = \frac{k^n}{\Gamma(n)} \left[-\log F(x)\right]^{n-1} \left[F(x)\right]^{k-1} f(x), \quad -\infty < x < \infty. \tag{1.5}$$

Prediction of future obsrvation, based on the observed progressively type-II censored sample arising from different lifetime models have been studied by many authors. Basak et al.(2006) considered the maximum likelihood predictors for different lifetime distributions based on progressively type-II right censored data. Viveros and Balakrishnan (1994) derived the prediction limits for future observations from exponential distribution based on progressively type-II censored data. Mousa and Jaheen (2002, 2005) discussed the Bayesian prediction for future observations on the on the basis of progressively type-II censored data when the underlying model is is Burr distribution and Rayleigh distribution. Panahi and Sayyareh (2014) discussed the parameter estimation and prediction of future order statistics for Burr type XII distribution. Some other works on this area are Guilbaud (2004), Balakrishnan et al. (2004) and Balakrishnan (2007).

In this paper, we discuss the problem of Bayesian prediction of future order statistics and lower k-record values from a future sample of observations arising from a generalized exponential (GE) distribution based on the observed progressively type-II censored sample. The pdf and cdf of GE distribution with parameters λ and α, denoted by GE(λ, α) are respectively given by

$$f(y) = \frac{\alpha}{\lambda}\left[1-\exp\left\{-\left(\frac{y}{\lambda}\right)\right\}\right]^{\alpha-1}\exp\left\{-\left(\frac{y}{\lambda}\right)\right\}, \quad 0 \le \mu < y < \infty, \lambda > 0, \alpha > 0. \tag{1.6}$$

and

$$F(y) = \left[1-\exp\left\{-\left(\frac{y}{\lambda}\right)\right\}\right]^{\alpha}, \ 0 \le \mu < y < \infty, \lambda > 0, \alpha > 0. \tag{1.7}$$

The GE distribution introduced by Gupta and Kundu (1999) can be used quite effectively in analyzing many lifetime data. When $\alpha = 1$, the distribution reduces to an exponential distribution with mean λ. The GE distribution has increasing or decreasing hazard rate depending on the shape parameter α. It is also observed that the GE model fits better than the gamma or the Weibull model in certain cases. Madi and Raqab (2009) developed the Bayesian inference of the parameters of GE distribution based on progressively type-II censored sample.

The paper is organized as follows. In Section 2, we explain the Bayesian estimation and obtain the prior and posterior distributions. In Section 3, we consder the Bayesian prediction of future order statistics and the corresponding prediction intervals based on the observed progressively type-II censored sample. The Bayesian point and interval

predictors for future lower k-record values are discussed in Section 4. Finally, a numerical study is considered in Section 5 to illustrate the performance of different predictors developed in this work and some concluding remarks are made in Section 6

2 Bayesian Estimation

In this section, we describe the Bayesian approach for the problem of inference and define the structure of prior and posterior distributions. In this approach , we consider both symmetric and asymmetric loss functions A symmetric loss function is the squared error loss (SEL) function which is defined as

$$L_1\left(\psi(\theta), \widehat{\psi}(\theta)\right) = \left(\psi(\theta) - \widehat{\psi}(\theta)\right)^2, \tag{2.1}$$

where $\widehat{\psi}(\theta)$ is an estimator of $\psi(\theta)$. The Bayes estimator of $\psi(\theta)$ under L_1 is defined as the mean of the posterior density function (2.1) given the informative sample $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ which is obtained as

$$\widehat{\psi}(\theta)_{BS} = E\left(\psi(\theta)|\mathbf{x}\right), \tag{2.2}$$

provided $E(\cdot)$ exists. An asymmetric loss function is the Linear Exponential (LINEX) loss (LL) function which is defined as

$$L_2\left(\psi(\theta), \widehat{\psi}(\theta)\right) = e^{h\left(\psi(\theta) - \widehat{\psi}(\theta)\right)} - h\left(\psi(\theta) - \widehat{\psi}(\theta)\right) - 1,\ h \neq 0. \tag{2.3}$$

The Bayes estimator of $\psi(\theta)$ under the loss function L_2 can be obtained as

$$\widehat{\psi}(\theta)_{BL} = -\frac{1}{h}\log\left[E\left(e^{-h\psi(\theta)}|\mathbf{x}\right)\right] \tag{2.4}$$

provided $E(\cdot)$ exists.

2.1 Prior and Posterior Distributions

Let $X_{1:m:n} \leq X_{2:m:n} \leq, \ldots, \leq X_{m:m:n}$, $(m \leq n)$ be a progressively type-II censored order statistics of size m with a censoring scheme $(R_1, R_2, \ldots, R_m)$ obtained from a life test on n items whose lifetimes follow GE(λ, α) distribution with pdf and cdf given in (1.6) and (1.7) respectively. Let $\mathbf{x}_{(\mathbf{m:n})} = (x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n})$ be the observed realization of the progressive sample $(X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n})$. Then by using (1.1), the likelihood

function of (λ, α) given $\mathbf{x_{(m:n)}}$ is obtained as

$$\begin{aligned} L(\lambda, \alpha|\mathbf{x_{(m:n)}}) &= C\left(\frac{\alpha}{\lambda}\right)^m \prod_{i=1}^{m} \left(\left[1-\exp\left\{-\left(\frac{x_{i:m:n}}{\lambda}\right)\right\}\right]^{\alpha-1} \exp\left\{-\left(\frac{x_{i:m:n}}{\lambda}\right)\right\}\right) \\ &\quad \times \prod_{i=1}^{m}\left[1-\left(1-\exp\left\{-\left(\frac{x_{i:m:n}}{\lambda}\right)\right\}\right)^{\alpha}\right]^{R_i}, \end{aligned} \tag{2.5}$$

where C is defined in (1.2). Since both scale and shape parameters are non-negative, we assume the prior distributions for λ and α as independent inverse gamma and gamma distributions as considered by Chacko and Muraleedharan (2018) and are respectively given by

$$\pi_1(\lambda|a,b) = \frac{b^a}{\Gamma a}\lambda^{-(a+1)}e^{-\frac{b}{\lambda}}, \quad \lambda > 0 \tag{2.6}$$

and

$$\pi_2(\alpha|c,d) = \frac{d^c}{\Gamma c}\alpha^{c-1}e^{-d\alpha}, \quad \alpha > 0 \tag{2.7}$$

where a, b, c and d are hyperparameters which are assumed to be known and non-negative. Then the joint prior density of (λ, α) is given by

$$\pi(\lambda,\alpha) \propto \exp\left\{-\left(d\alpha + \frac{b}{\lambda}\right)\right\}\alpha^{c-1}\lambda^{-(a+1)}, \quad \lambda > 0,\ \alpha > 0. \tag{2.8}$$

Thus the joint posterior density function of (λ, α) given the observed data $\mathbf{x_{(m:n)}}$ is obtained as

$$\pi^*(\lambda,\alpha|\mathbf{x_{(m:n)}}) = \frac{L(\lambda,\alpha|\mathbf{x_{(m:n)}})\pi(\lambda,\alpha)}{\int_{\alpha=0}^{\infty}\int_{\lambda=0}^{\infty} L(\lambda,\alpha|\mathbf{x_{(m:n)}})\pi(\lambda,\alpha)d\lambda d\alpha}, \tag{2.9}$$

where $\pi(\lambda, \alpha)$ is defined in (2.8). Then the expected value of any function $\psi(\lambda, \alpha)$ of λ and α with respect to (2.9) is given by

$$E_{\pi^*}\left[\psi(\lambda,\alpha)\right] = \frac{\int_{\alpha=0}^{\infty}\int_{\lambda=0}^{\infty}\psi(\lambda,\alpha)\, L(\lambda,\alpha|\mathbf{x_{(m:n)}})\pi(\lambda,\alpha)d\lambda d\alpha}{\int_{\alpha=0}^{\infty}\int_{\lambda=0}^{\infty} L(\lambda,\alpha|\mathbf{x_{(m:n)}})\pi(\lambda,\alpha)d\lambda d\alpha}. \tag{2.10}$$

Since it is not possible to get an explicit expression for (2.9), we propose importance sampling method to obtain independent sample observations for the Bayesian prediction.

2.2 Importance Sampling method

In this subsection, we consider the importance sampling method to generate samples from the posterior distribution (2.9) and then find the Bayes predictors under different

loss functions. By using (2.5) and (2.8), the numerator of (2.9) can be expressed as

$$\begin{aligned} L(\lambda,\alpha|\mathbf{x_{(m:n)}})\pi(\lambda,\alpha) \quad \propto \quad & \alpha^{m+c-1}\lambda^{-(a+m+1)}\exp\left\{-\alpha\left[d-\sum_{i=1}^{m}\log\left(1-e^{\frac{x_{i:m:n}}{\lambda}}\right)\right]\right\} \\ & \times\exp\left\{-\frac{(b+x_{i:m:n})}{\lambda}\right\}\exp\left\{-\sum_{i=1}^{m}\log\left(1-e^{\frac{x_{i:m:n}}{\lambda}}\right)\right\} \\ & \times\prod_{i=1}^{s}\left[1-\left(1-\exp\left\{-\left(\frac{x_{i:m:n}}{\lambda}\right)\right\}\right)^{\alpha}\right]^{R_i}. \end{aligned} \tag{2.11}$$

Define

$$\bar{x}=\frac{1}{m}\sum_{i=1}^{m}x_{i:m:n}\ , \quad t(\lambda)=-\sum_{i=1}^{m}\log\left(1-e^{\frac{x_{i:m:n}}{\lambda}}\right)$$

and

$$R(\lambda,\alpha)=\sum_{i=1}^{m}r_i\log\left[1-\left(1-\exp\left\{-\left(\frac{x_{i:m:n}}{\lambda}\right)\right\}\right)^{\alpha}\right].$$

Now (2.9) can be expressed as

$$\pi^{*}(\lambda,\alpha|\mathbf{x_{(m:n)}})\propto g_1(\lambda)\,g_2(\alpha|\lambda)w(\lambda,\alpha),$$

where

$$g_1(\lambda)\propto\lambda^{-(a+m+1)}\exp\left\{-\frac{(b+m\bar{x})}{\lambda}\right\}, \tag{2.12}$$

$$g_2(\alpha|\lambda)\propto[d+t(\lambda)]^{(m+c)}\,\alpha^{m+c-1}\exp\{-\alpha[d+t(\lambda)]\} \tag{2.13}$$

and

$$w(\lambda,\alpha)=\frac{\exp\{t(\lambda)+R(\lambda,\alpha)\}}{[d+t(\lambda)]^{(m+a)}}. \tag{2.14}$$

From (2.12), we can see that λ follows an inverse gamma distribution with parameters $(b+m\bar{x})$ and $(a+m)$. Therefore we can easily generate random sample of observations from the distribution of λ given in (2.12). For a given value of λ, (2.13) shows that α follows a gamma distribution with parameters $(m+c)$ and $[d+t(\lambda)]$. Then the expected value of any function $\psi(\lambda,\alpha)$ of λ and α given in (2.10) can be expressed as

$$\begin{aligned} E_{\pi^*}[\psi(\lambda,\alpha)] &= \frac{\int_{\alpha=0}^{\infty}\int_{\lambda=0}^{\infty}\psi(\lambda,\alpha)\,g_1(\lambda)\,g_2(\alpha|\lambda)w(\lambda,\alpha)\,d\lambda d\alpha}{\int_{\alpha=0}^{\infty}\int_{\lambda=0}^{\infty}g_1(\lambda)\,g_2(\alpha|\lambda)w(\lambda,\alpha)\,d\lambda d\alpha} \\ &= \frac{E_g[\psi(\lambda,\alpha)\,w(\lambda,\alpha)]}{E_g[w(\lambda,\alpha)]}, \end{aligned} \tag{2.15}$$

where $E_g[.]$ denote the expectation with respect to the pdf given by

$$g(\lambda,\alpha)=g_1(\lambda)\,g_2(\alpha|\lambda). \tag{2.16}$$

Let $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)}$ be N observations generated from (2.12) and let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}$ be N observations generated from (2.13). Then by using importance sampling method, an approximated value of (2.15) is given by

$$E_{\pi^*}\left[\psi\left(\lambda,\alpha\right)\right] = \frac{\sum_{i=1}^{N} \psi\left(\lambda^{(i)},\alpha^{(i)}\right) w\left(\lambda^{(i)},\alpha^{(i)}\right)}{\sum_{i=1}^{N} w\left(\lambda^{(i)},\alpha^{(i)}\right)}, \tag{2.17}$$

where $w\left(\cdot,\cdot\right)$ is defined in (2.14).

3 Bayesian Prediction

3.1 Bayesian point prediction of order statistics from a future sample

In this subsection, we consider the two-sample Bayesian point prediction of future order statistics based on the observed progressively type-II censored sample. Recently, this problem has been received considerable attension in statistical inference. For more extensive discussion on this area, one may refer Kundu and Raqab (2012).

Let $X_{1:m:n} \leq X_{2:m:n} \leq, \ldots, \leq X_{m:m:n}$, $(m \leq n)$ be the observed progressively type-II censored order statistics with a pre-fixed censoring scheme $(R_1, R_2, \ldots, R_m)$. Let $Z_1, Z_2, \ldots, Z_s$ be a future random sample of size s arising from GE(λ, α) distribution. Let $Z_{1:s} \leq Z_{2:s} \leq \cdots \leq Z_{s:s}$ be the order statistics corresponding to the sample observations. Then by using (1.4), the density function of $Z_{r:s}$ is obtained as

$$g_{r:s}(z|\lambda,\alpha) = \frac{\alpha e^{-\frac{z}{\lambda}}}{\lambda B(r, s-r+1)} \left[1-\left(1-e^{-\frac{z}{\lambda}}\right)^{\alpha}\right]^{(s-r)} \left(1-e^{-\frac{z}{\lambda}}\right)^{\alpha r-1}, \ \alpha,\lambda>0, z>0. \tag{3.1}$$

The Bayesian predictive density function of $Z_{r:s}$, based on the observed data $\mathbf{x}_{(\mathbf{m:n})}$ is given by

$$\widehat{g}_{r:s}(z|\mathbf{x}_{(\mathbf{m:n})}) = \int_{\alpha=0}^{\infty}\int_{\lambda=0}^{\infty} g_{r:s}(z|\lambda,\alpha)\pi^*(\lambda,\alpha|\mathbf{x}_{(\mathbf{m:n})})d\lambda d\alpha, \tag{3.2}$$

where $\pi^*(\lambda,\alpha|\mathbf{x}_{(\mathbf{m:n})})$ is defined in (2.9). The Bayesian point predictor of $Z_{r:s}$ based on the observed data $\mathbf{x}_{(\mathbf{m:n})}$ under the SEL function is obtained by the mean of the predictive density function (3.2) which is given by

$$\begin{aligned}
\widehat{Z}_{r:s}^{(BS)} &= E_{\widehat{g}_{r:s}}[Z_{r:s}|\mathbf{x}_{(\mathbf{m:n})}] \\
&= \int_{z=0}^{\infty} z\widehat{g}_{r:s}(z|\mathbf{x}_{(\mathbf{m:n})})dz \\
&= \int_{\alpha=0}^{\infty}\int_{\lambda=0}^{\infty} A_{r:s}\left(\lambda,\alpha\right)\pi^*(\lambda,\alpha|\mathbf{x}_{(\mathbf{m:n})})d\lambda d\alpha \\
&= E_{\pi^*}\left[A_{r:s}\left(\lambda,\alpha\right)\right],
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned} A_{r:s}(\lambda,\alpha) &= \int_{z=0}^{\infty} z g_{r:s}(z|\mathbf{x}_{(\mathbf{m}:\mathbf{n})})dz \\ &= \frac{\alpha}{\lambda B(r,s-r+1))}\int_{z=0}^{\infty} z e^{-\frac{z}{\lambda}}\left[1-\left(1-e^{-\frac{z}{\lambda}}\right)^{\alpha}\right]^{(s-r)}\left(1-e^{-\frac{z}{\lambda}}\right)^{\alpha r-1}dz \\ &= E_T\left[-\lambda\log\left(1-T^{\frac{1}{\alpha}}\right)\right], \end{aligned} \tag{3.4}$$

where T follows a beta type I distribution with density function given by

$$f_1^*(t) = \frac{1}{B(r,s-r+1))}t^{r-1}(1-t)^{s-r},\ 0<t<1. \tag{3.5}$$

Now by using (2.17), the Bayesian point predictor of $Z_{r:s}$, under the SEL function is obtained as

$$\widehat{Z}_{r:s}^{(BS)} = \frac{\sum_{i=1}^{N} A_{r:s}\left(\lambda^{(i)},\alpha^{(i)}\right) w\left(\lambda^{(i)},\alpha^{(i)}\right)}{\sum_{i=1}^{N} w\left(\lambda^{(i)},\alpha^{(i)}\right)}, \tag{3.6}$$

where $w(\cdot,\cdot)$ is defined in (2.14) and $A_{r:s}(\cdot,\cdot)$ is defined in (3.4). The Bayesian point predictor of $Z_{r:s}$, under the LL function is obtained as

$$\begin{aligned} \widehat{Z}_{r:s}^{(BL)} &= -\frac{1}{c}\log E_{\widehat{g}_{r:s}}[e^{-cZ_{r:s}}|\mathbf{x}_{(\mathbf{m}:\mathbf{n})}],\quad c>0 \\ &= -\frac{1}{c}\log\left[\int_{z=0}^{\infty} e^{-cz}\widehat{g}_{r:s}(z|\mathbf{x}_{(\mathbf{m}:\mathbf{n})})dz\right] \\ &= -\frac{1}{c}\log\left[\int_{\alpha=0}^{\infty}\int_{\lambda=0}^{\infty} B_{r:s}(\lambda,\alpha)\,\pi^*\left(\lambda,\alpha|\mathbf{x}_{(\mathbf{m}:\mathbf{n})}\right)d\lambda d\alpha\right] \\ &= -\frac{1}{c}\log E_{\pi^*}\left[B_{r:s}(\lambda,\alpha)\right], \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} B_{r:s}(\lambda,\alpha) &= \int_{z=0}^{\infty} e^{-cz} g_{r:s}(z|\mathbf{x}_{(\mathbf{m}:\mathbf{n})})dz \\ &= \frac{\alpha}{\lambda B(r,s-r+1))}\int_{z=0}^{\infty} e^{-z\left(c+\frac{1}{\lambda}\right)}\left[1-\left(1-e^{-\frac{z}{\lambda}}\right)^{\alpha}\right]^{(s-r)}\left(1-e^{-\frac{z}{\lambda}}\right)^{\alpha r-1}dz \\ &= E_T\left[\left(1-T^{\frac{1}{\alpha}}\right)^{1+c\lambda}\right], \end{aligned} \tag{3.8}$$

where T has the density function given in (3.5). By using (2.17), the Bayesian point predictor of $Z_{r:s}$, under the LL function is

$$\hat{Z}_{r:s}^{(BL)} = -\frac{1}{c}\log\left[\frac{\sum_{i=1}^{N} B_{r:s}\left(\lambda^{(i)},\alpha^{(i)}\right) w\left(\lambda^{(i)},\alpha^{(i)}\right)}{\sum_{i=1}^{N} w\left(\lambda^{(i)},\alpha^{(i)}\right)},\right], \tag{3.9}$$

where $w(\cdot,\cdot)$ is defined in (2.14) and $B_{r:s}(\cdot,\cdot)$ is defined in (3.8).

3.2 Prediction intervals for future order statistics

In this subsection, we construct the $100(1-\gamma)\%$ prediction interval for $Z_{r:s}$. The Bayesian predictive cdf of $Z_{r:s}$ based on the observed data $\mathbf{x}_{(\mathbf{m}:\mathbf{n})}$ is given by

$$\begin{aligned}\widehat{F}_{r:s}(z|\mathbf{x}_{(\mathbf{m}:\mathbf{n})}) &= \int_{y=0}^{z} \widehat{f}_{r:s}(y|\mathbf{x}_{(\mathbf{m}:\mathbf{n})})dy \\ &= E_{\pi^*}\left[G\left(1-e^{-\frac{z}{\lambda}}\right)^{\alpha}\right], \end{aligned} \tag{3.10}$$

where $G(\cdot)$ denote the cdf of T and $\pi^*(\lambda,\alpha|\mathbf{x}_{(\mathbf{m}:\mathbf{n})})$ is defined in (2.9). Hence by using (2.17), (3.10) can be approximated as

$$\widehat{F}_{r:s}(z|\mathbf{x}_{(\mathbf{m}:\mathbf{n})}) = \frac{\sum_{i=1}^{N} G\left(1-e^{-\frac{z}{\lambda^{(i)}}}\right)^{\alpha^{(i)}} w\left(\lambda^{(i)},\alpha^{(i)}\right)}{\sum_{i=1}^{N} w\left(\lambda^{(i)},\alpha^{(i)}\right)},$$

where $w(\cdot,\cdot)$ is defined in (2.14).

Now the two sided equi-tailed prediction interval is defined as (L,U), where L and U are obtained by solving the following non-linear equations

$$\widehat{F}_{r:s}(L|\mathbf{x}_{(\mathbf{m}:\mathbf{n})}) = \frac{\gamma}{2} \quad \text{and} \quad \widehat{F}_{r:s}(U|\mathbf{x}_{(\mathbf{m}:\mathbf{n})}) = 1-\frac{\gamma}{2}. \tag{3.11}$$

4 Bayesian point prediction of future lower *k*-record values

In this section, we obtain the Bayesian two-sample point prediction for the rth lower k-record value $L_{r(k)}$ from a future sample based on the observed progressively type II censored data $\mathbf{x}_{(\mathbf{m}:\mathbf{n})}$ under the SEL and LL functions.

Let $Y_1, Y_2, \ldots$ be a future sequence of random variables arising from $\mathrm{GE}(\lambda,\alpha)$ distribution and let $L_{r(k)}$ be the rth lower k-record value extracted from the sequence $Y_1, Y_2, \ldots$. Then by using (1.7) and (1.6) in (1.5), the pdf of $L_{r(k)}$ is obtained as

$$h_{r(k)}(y|\lambda,\alpha) = \frac{k^r\alpha^r e^{-\frac{y}{\lambda}}}{\lambda\Gamma(r)}\left[-\log\left(1-e^{-\frac{y}{\lambda}}\right)\right]^{r-1}\left(1-e^{-\frac{y}{\lambda}}\right)^{\alpha k-1},\ y>0,\alpha,\lambda>0. \tag{4.1}$$

The Bayesian predictive density function of $L_{r(k)}$, based on the observed data $\mathbf{x}_{(\mathbf{m}:\mathbf{n})}$ is given by

$$\widehat{h}_{r(k)}(y|\mathbf{x}_{(\mathbf{m}:\mathbf{n})}) = \int_{\alpha=0}^{\infty}\int_{\lambda=0}^{\infty} h_{r(k)}(y|\lambda,\alpha)\pi^*(\lambda,\alpha|\mathbf{x}_{(\mathbf{m}:\mathbf{n})})d\lambda d\alpha, \tag{4.2}$$

where $\pi^*(\lambda,\alpha|\mathbf{x}_{(\mathbf{m:n})})$ is defined in (2.9). Then the Bayesian point predictor of $L_{r(k)}$, under the SEL function is obtained as the mean of the predictive density function (4.2) and is given by

$$\begin{aligned}\widehat{L}_{r(k)}^{(BS)} &= E_{\widehat{h}_{r(k)}}[L_{r(k)}|\mathbf{x}_{(\mathbf{m:n})}] \\ &= \int_{y=0}^{\infty} y\widehat{h}_{r(k)}(y|\mathbf{x}_{(\mathbf{m:n})})dy \\ &= \int_{\alpha=0}^{\infty}\int_{\lambda=0}^{\infty} C_r(\lambda,\alpha)\,\pi^*(\lambda,\alpha|\mathbf{x}_{(\mathbf{m:n})})d\lambda d\alpha \\ &= E_{\pi^*}\left[C_r(\lambda,\alpha)\right], \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} C_r(\lambda,\alpha) &= \int_{y=0}^{\infty} y h_{r(k)}(y|\lambda,\alpha)dy \\ &= \frac{k^r\alpha^r}{\lambda\Gamma(r)}\int_{y=0}^{\infty} y e^{-\frac{y}{\lambda}}\left[-\log\left(1-e^{-\frac{y}{\lambda}}\right)\right]^{r-1}\left(1-e^{-\frac{y}{\lambda}}\right)^{\alpha k-1}dy \\ &= E_W\left[-\lambda\log\left(1-e^{-W}\right)\right], \end{aligned} \tag{4.4}$$

where W follows a gamma distribution with pdf given by

$$f_2^*(w) = \frac{(\alpha k)^r}{\Gamma(r)}e^{-\alpha k w}w^{r-1},\ 0<w<\infty. \tag{4.5}$$

Now by using (2.17), the Bayesian point predictor of $L_{r(k)}$, under the SEL function is obtained as

$$\widehat{L}_{r(k)}^{(BS)} = \frac{\sum_{i=1}^{N} C_r\left(\lambda^{(i)},\alpha^{(i)}\right)w\left(\lambda^{(i)},\alpha^{(i)}\right)}{\sum_{i=1}^{N} w\left(\lambda^{(i)},\alpha^{(i)}\right)}, \tag{4.6}$$

where $w(\cdot,\cdot)$ is defined in (2.14) and $C_r(\cdot,\cdot)$ is defined in (4.4). The Bayesian point predictor of $L_{r(k)}$ under the LL function is obtained as

$$\begin{aligned}\widehat{L}_{r(k)}^{(BL)} &= -\frac{1}{c}\log E_{\widehat{f}^*_{r(k)}}[e^{-cL_{r(k)}}|\mathbf{x}_{(\mathbf{m:n})}] \\ &= -\frac{1}{c}\log\left(\int_{y=0}^{\infty} e^{-cy}\widehat{f}^*_{r(k)}(y|\mathbf{x}_{(\mathbf{m:n})})dy\right) \\ &= -\frac{1}{c}\log\left[\int_{\alpha=0}^{\infty}\int_{\lambda=0}^{\infty} D_r(\lambda,\alpha)\,\pi^*(\lambda,\alpha|\mathbf{x}_{(\mathbf{m:n})})d\lambda d\alpha\right] \\ &= -\frac{1}{c}\log E_{\pi^*}\left[D_r(\lambda,\alpha)\right], \end{aligned} \tag{4.7}$$

where $\pi^*(\lambda,\alpha|\mathbf{x_{(m:n)}})$ is defined in (2.9) and

$$\begin{aligned} D_r(\alpha,\lambda) &= \int_{y=0}^{\infty} e^{-cy} f^*_{r(k)}(y|\lambda,\alpha)dy \\ &= \frac{k^r\alpha^r}{\lambda\Gamma(r)}\int_{y=0}^{\infty} e^{-y\left(c+\frac{1}{\lambda}\right)}\left[-\log\left(1-e^{-\frac{y}{\lambda}}\right)\right]^{r-1}\left(1-e^{-\frac{y}{\lambda}}\right)^{\alpha k-1}dy \\ &= E_W\left[\left(1-e^{-W}\right)^{1+c\lambda}\right], \end{aligned} \tag{4.8}$$

where W follows a gamma distribution with pdf given in (4.5). By using (2.17), the Bayesian point predictor of $L_{r(k)}$, under the LL function is obtained as

$$\hat{L}^{(BL)}_{r(k)} = -\frac{1}{c}\log\left[\frac{\sum_{i=1}^{N} D_r\left(\lambda^{(i)},\alpha^{(i)}\right) w\left(\lambda^{(i)},\alpha^{(i)}\right)}{\sum_{i=1}^{N} w\left(\lambda^{(i)},\alpha^{(i)}\right)}\right], \tag{4.9}$$

where $w(\cdot,\cdot)$ is defined in (2.14) and $D_r(\cdot,\cdot)$ is defined in (4.8).

4.1 Prediction intervals for future Lower *k*-record values

In this subsection, we construct the $100(1-\gamma)\%$ prediction interval for $L_{r(k)}$. The Bayesian predictive cdf of $L_{r(k)}$ based on the observed data $\mathbf{x_{(m:n)}}$ is given by

$$\begin{aligned} \widehat{F^*}_{r(k)}(z|\mathbf{x_{(m:n)}}) &= \int_{y=0}^{z} \widehat{f^*}_{r(k)}(y|\mathbf{x_{(m:n)}})dy \\ &= E_{\pi^*}\left[\bar{H}\left(-\log\left(1-e^{-\frac{z}{\lambda}}\right)\right)\right], \end{aligned} \tag{4.10}$$

where $\bar{H}(\cdot)$ denote the survival function of W and $\pi^*(\lambda,\alpha|\mathbf{x_{(m:n)}})$ is defined in (2.9). Hence by using (2.17), (4.10) can be approximated by

$$\widehat{F^*}_{r(k)}(z|\mathbf{x_{(m:n)}}) = \frac{\sum_{i=1}^{N}\bar{H}\left[-\log\left(1-e^{-\frac{z}{\lambda^{(i)}}}\right)\right] w_1\left(\lambda^{(i)},\alpha^{(i)}\right)}{\sum_{i=1}^{N} w\left(\lambda^{(i)},\alpha^{(i)}\right)}.$$

Now the two sided equi-tailed prediction interval is defined as (L,U), where L and U are obtained by solving the following non-linear equations

$$\widehat{F}_{r(k)}(L|\mathbf{x_{(m:n)}}) = \frac{\gamma}{2} \quad \text{and} \quad \widehat{F}_{r(k)}(U|\mathbf{x_{(m:n)}}) = 1-\frac{\gamma}{2}. \tag{4.11}$$

5 Numerical Illustration

Example 5.1. *Simulated Data*

Here we consider two different sampling schemes with $n = 30$ and $m = 25$ to generate the progressively type-II censored samples, namely:
Scheme I : $R = \left(0^{(10)}, 1^{(5)}, 0^{(10)}\right)$
Scheme II : $R = \left(0^{(10)}, 2^{(1)}, 3^{(3)}, 5^{(1)}, 0^{(10)}\right)$,
where $a^{(i)}$ denotes the positive integer a repeating i times. The point predictors and interval predictors for future order statistics and lower k-records are obtained using importance sampling approach, which assumes the following steps. We took the hyper parameters for prior distributions as $a = b = c = d = 2$ and the true values for the parameters are selected as: $(\lambda, \alpha) = (1, 1.5), (1, 1)$ and $(2, 1)$. The algorithm for obtaining Bayes point predictors for future order statistics and lower k-record values is given below.

1. For a given n, generate m progressive type-II censored sample from $\mathrm{GE}(\lambda, \alpha)$
2. Set t=1
3. Generate $\lambda^{(t)}$ from Inverse gamma distribution with parameters $(b + m\bar{x})$ and $(a + m)$
4. For each $\lambda^{(t)}$, generate $\alpha^{(t)}$ from gamma distribution with parameters $(m + c)$ and $\left(d + t(\lambda^{(t)})\right)$
5. Set $t = t + 1$
6. Repeat the steps (2) to (4) for $N = 10000$ times
7. Compute (3.6), (3.9), (4.6), (4.9) using the simulated sample $\left(\lambda^{(1)}, \alpha^{(1)}\right), \left(\lambda^{(2)}, \alpha^{(2)}\right) \ldots, \left(\lambda^{(N)}, \alpha^{(N)}\right)$.

The Bayesian predictive estimates of future order statistics $Z_{r:s},\ r = 1, 2 \ldots, s$ for size $s = 12$ and $s = 15$ based on the observed censored data under the SEL and LL functions and its prediction intervals for Scheme I and Scheme II are evaluated and presented in Table 1.

The Bayesian predictive estmates of future lower k-record $L_{r(k)}$ for $k = 2$ and $k = 3$ based on the observed censored data under the SEL and LL functions and its prediction intervals for Scheme I and Scheme II are presented in Table 2.

Example 5.2. *Real-life Data*

The followig data represents the distance to failure for 38 vehicle shock absorbers. The data set is taken from Meeker and Escobar (2014). The corresponding observations

are
67.0, 69.5, 78.2, 87.9, 91.2, 96.6, 98.2, 113.1, 116.9, 118.5, 118.8, 121.4, 122.0, 128.7, 131.5, 133.3, 134.7, 140.4, 143.0, 175.2, 175.4, 178.9, 184.5, 189.6, 189.8, 194.1, 201.0, 201.0, 201.5, 203.2, 209.0, 227.0, 234.9, 265.1, 274.1, 274.9, 278.9, 281.0.
Singh et al.(2020) obtained that the Kolmogorov– Smirnov distance is 0.1201 and the corresponding p value is 0.6434 for $\lambda = 77.51$ and $\alpha = 4.5264$. Therefore, the GE distribution provides a reasonable fit for the data set. We have generated progressively type-II censored samples using the following sampling schemes.

n	m	Scheme	Progressively type II censored Data
38	10	$(0^{(4)}, 1^{(2)}, 0^{(4)})$	67.0, 69.5, 78.2, 87.9, 91.2, 96.6, 98.2, 116.9, 118.5, 121.4
38	12	$(0^{(4)}, 1^{(2)}, 3^{(2)}, 5, 0^{(3)})$	67.0,69.5, 78.2, 87.9,91.2, 96.6, 98.2, 116.9,122.0, 134.7, 184.5, 189.6

The Bayesian predictive estimates of future order statistics $Z_{r:s},\ r = 1, 2 \ldots, s$ of size $s = 12$ based on the observed censored data under the SEL and LL functions and its prediction intervals for Scheme I and Scheme II are presented in Table 3.

The Bayesian predictive estimates of future lower k-record value $L_{r(k)}$ for $k = 2$ and $k = 3$ based on the observed censored data under the SEL and LL functions and its prediction intervals for Scheme I and Scheme II are presented in Table 4.

6 Conclusion

The two-sample Bayesian prediction of order statistics and lower k-record values from a future sample based on observed progressively type-II censored sample arising from a GE distribution are considered in this work. Importance sampling method has been used to develop the Bayesian point and interval predictors under symmetric and asymmetric loss functions. A numerical study is also considered to illustrate the inferential procedures developed in this paper.

Table 1: Prediction values and intervals of $Z_{r:s}$ for $s = 10$ and $s = 15$.

(λ,α,s)	r	Scheme I				Scheme II			
		$\widehat{Z}_{r:s}^{(BS)}$	$\widehat{Z}_{r:s}^{(BL)}$	L	U	$\widehat{Z}_{r:s}^{(BS)}$	$\widehat{Z}_{r:s}^{(BL)}$	L	U
(1.5,1,10)	1	0.05247	0.04198	0.02715	0.52691	0.04242	0.03542	0.02972	0.51769
	2	0.14085	0.13120	0.07360	0.68185	0.09323	0.08637	0.08499	0.60038
	3	0.25189	0.23211	0.13773	0.84958	0.19448	0.17671	0.15011	0.75151
	4	0.3906	0.35620	0.20723	1.02127	0.34136	0.30412	0.21926	0.91285
	5	0.56069	0.50585	0.28266	1.20726	0.54555	0.47595	0.29474	1.09792
	6	0.76858	0.68528	0.36571	1.41797	0.82623	0.70386	0.38023	1.32601
	7	1.02462	0.90120	0.45913	1.66821	1.21744	1.00730	0.48206	1.63494
	8	1.34601	1.16420	0.56753	1.98324	1.78794	1.42169	0.61351	2.12605
	9	1.76334	1.49197	0.69913	2.41551	2.71043	2.02217	0.81483	3.26576
	10	2.33832	1.91675	0.87189	3.10680	2.94570	2.10580	1.02560	4.15781
(1.5,1,15)	1	0.04514	0.04874	0.01478	0.20145	0.03256	0.03588	0.02581	0.15870
	2	0.08375	0.07948	0.06674	0.56114	0.02800	0.02711	0.01998	0.31226
	3	0.15491	0.14562	0.11915	0.67975	0.05661	0.05413	0.04344	0.39968
	4	0.24530	0.22856	0.17354	0.79502	0.09639	0.09181	0.07102	0.48695
	5	0.35623	0.32902	0.23038	0.91293	0.14889	0.14069	0.10202	0.57783
	6	0.49003	0.44854	0.29034	1.03783	0.21612	0.20260	0.13642	0.67526
	7	0.65025	0.58949	0.35450	1.17407	0.30078	0.27965	0.17453	0.78251
	8	0.84207	0.75531	0.42401	1.32694	0.40663	0.37479	0.21715	0.90368
	9	1.07320	0.95099	0.50079	1.50410	0.53904	0.49211	0.26532	1.04475
	10	1.35545	1.18385	0.58742	1.71758	0.70614	0.63763	0.32088	1.21537
	11	1.70792	1.46514	0.68813	1.98937	0.92096	0.82059	0.38679	1.43293
	12	2.16430	1.81326	0.81068	2.36558	1.20637	1.05632	0.46867	1.73411
	13	2.79218	2.26146	0.97162	2.97230	1.60817	1.37308	0.57853	2.21857
	14	3.76087	2.88052	1.22190	4.39196	2.24101	1.83264	0.75423	3.34484
	15	4.50189	4.01553	1.96553	5.09981	3.56007	3.02571	0.79981	4.52001

Table 2: Prediction values and intervals of $L_{r(k)}$ for the Scheme I and Scheme II

k	(λ,α)	r	*Scheme I*				*Scheme II*			
			$\widehat{L}_{r(k)}^{(BS)}$	$\widehat{L}_{r(k)}^{(BL)}$	L	U	$\widehat{L}_{r(k)}^{(BS)}$	$\widehat{L}_{r(k)}^{(BL)}$	L	U
2	(1.5,1)	1	2.03153	1.08928	0.24773	2.50437	2.13753	1.85279	0.48773	2.81437
		2	0.71021	0.48514	0.13798	1.40623	1.03311	0.70084	1.96564	1.96564
		3	0.30209	0.23415	0.08540	0.96798	0.43222	0.33288	1.22466	1.22466
		4	0.13804	0.11599	0.05522	0.71798	0.21903	0.18189	0.90824	0.90824
		5	0.06534	0.05799	0.03662	0.55338	0.11518	0.10079	0.70057	0.70057
	(1,1)	1	1.34766	0.80439	0.13576	2.09508	1.94766	1.48044	0.53576	2.91783
		2	0.44390	0.32773	0.06330	1.12164	0.83330	0.59687	1.81463	1.81463
		3	0.17853	0.14672	0.03330	0.73857	0.29762	0.24057	1.01284	1.01284
		4	0.07727	0.06785	0.01846	0.52447	0.14705	0.12671	0.72495	0.72495
		5	0.03470	0.03181	0.01052	0.38701	0.07551	0.06796	0.53973	0.53973
3	(1.5,1)	1	2.60697	1.46525	0.37933	2.70812	2.97014	1.89553	2.79335	2.71186
		2	1.09484	0.75934	0.24349	1.60451	1.51636	1.04977	2.24386	2.24386
		3	0.55459	0.42803	0.17110	1.15774	0.74863	0.57686	1.46778	1.46778
		4	0.30133	0.24930	0.12509	0.89790	0.44073	0.36193	1.13771	1.13771
		5	0.16964	0.14744	0.09352	0.72304	0.26903	0.23125	0.91612	0.91612
	(1,1)	1	1.74367	1.08064	0.23155	2.27921	1.94367	1.06415	2.92316	2.75421
		2	0.69718	0.51969	0.13195	1.29935	1.22599	0.89411	2.09213	2.09213
		3	0.33744	0.27561	0.08345	0.90606	0.52332	0.42231	1.23977	1.23977
		4	0.17544	0.15185	0.05519	0.68040	0.30181	0.25719	0.93575	0.93575
		5	0.09460	0.08524	0.03745	0.53090	0.18065	0.15999	0.73429	0.73429

Table 3: Prediction values and intervals of $Z_{r:s}$ for $s = 12$ from the real data set

r	Scheme I				Scheme II			
	$\widehat{Z}_{r:s}^{(BS)}$	$\widehat{Z}_{r:s}^{(BL)}$	L	U	$\widehat{Z}_{r:s}^{(BS)}$	$\widehat{Z}_{r:s}^{(BL)}$	L	U
1	40.0146	30.8103	26.8214	55.3025	39.2574	35.9234	34.2891	66.9578
2	45.4100	36.0150	31.7784	61.5647	44.6224	38.8479	37.3953	73.5607
3	50.2472	41.7532	36.5225	67.0218	49.0541	45.6822	43.2307	81.4904
4	54.5136	47.2793	40.4324	73.0790	52.9551	49.0316	48.0186	88.9323
5	58.6073	53.5426	44.0144	80.1587	56.6933	54.0125	52.3806	94.4059
6	62.7563	58.9826	47.5094	86.5649	60.4780	59.0113	56.6122	98.9863
7	67.1544	64.6272	51.0827	91.6198	64.4867	63.5742	60.9145	106.9711
8	71.9098	69.5121	54.8919	98.7710	68.9275	66.2626	65.4771	113.0128
9	77.7277	75.6878	59.1376	105.7811	74.1140	71.1832	70.5389	126.3519
10	84.7880	80.9225	64.1442	112.2324	80.6162	77.4294	76.4846	161.9368
11	95.1064	86.2454	70.5820	118.4385	89.9208	85.9245	84.1114	170.0663
12	103.2157	92.3701	76.2158	124.2351	98.2371	94.2360	93.6981	178.1892

Table 4: Prediction values and intervals of $L_{r(k)}$ from the real data set

k	r	Scheme I				Scheme II			
		$\widehat{L}_{r(k)}^{(BS)}$	$\widehat{L}_{r(k)}^{(BL)}$	L	U	$\widehat{L}_{r(k)}^{(BS)}$	$\widehat{L}_{r(k)}^{(BL)}$	L	U
2	1	60.2347	42.1820	39.4278	74.2314	61.2389	50.2354	49.0258	81.3640
	2	57.2846	39.2561	37.0127	69.2573	56.9524	45.3287	44.9102	70.1270
	3	53.6011	35.4619	33.9089	62.4979	52.1094	41.1237	39.6014	65.7585
	4	47.5775	33.8280	30.2881	60.2247	46.5996	36.3426	35.3000	62.6641
	5	43.1134	29.4673	27.4546	55.6120	42.5069	32.8130	31.9430	58.0706
3	1	72.2691	53.6912	47.2050	80.2578	66.3205	58.9624	57.2018	84.2089
	2	66.0248	50.9851	44.2389	76.3257	62.3287	54.2028	52.9364	79.3624
	3	60.9749	46.2989	41.0514	72.0442	58.8461	49.3268	48.2526	75.5336
	4	54.8793	42.5678	37.2857	62.7478	53.2829	31.7031	43.7754	65.4027
	5	50.3438	39.9639	34.3216	58.0899	49.1373	30.6562	40.2587	61.7469

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